IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

# A COMPARISON OF BIASED REDUCED MODELS

**Rupesh Kumari** *
M.Tech. (IIT, Roorkee)

## ABSTRACT

In this paper a comparison of some biased reduced order models is carried out. These methods use the advantages of Pade approximation, stability equation method, continued fraction expansion method based on first and second cauer forms, factor division algorithm, and Routh approximation. One numerical example illustrates the method.

**KEYWORDS**: Stability, Biased Models, Model Reduction.

## INTRODUCTION

Model order reduction means to reduce a high order system into a low order model which retains the desired characteristics of the system. A large number of research papers on model order reduction have been published [1-4]. Some parameters are changed to get various reduced order models. In this paper four methods are discussed to produce reduced order models. The first method is based on stability equation method and a combination of time moments and Markov parameters [5-6]. The second method utilises the advantages of continued fraction expansion and time moments and Markov parameters [7]. In the third method factor division is coupled with time moments and Markov parameters [8-9]. FourthSimplified RouthApproximation Method is coupled with the combination of time moments and Markov parameters and there is no need of solving the Pade equations to calculate reduced numerator [10].Some biased methods are extended for order reduction of multivariable system as in Improved Pade Approximants Using Stability Equation is extended for multivariable system. In this paper comparison of biased reduced models is carried out.

## STATEMENT OF PROBLEM

Let higher order system may be described by the transfer function

$$G(s) = \frac{A_{21} + A_{22}\,s + A_{23}s^2 + \cdots \ldots\ldots A_{2n}s^{n-1}}{A_{11} + A_{12}\,s + A_{13}s^2 + \cdots \ldots\ldots + A_{1,n+1}s^n} \quad \ldots\ldots\ldots (1)$$

Objective is to reduce the above $n^{th}$ order system to $r^{th}$ order reduced model which is defined by the transfer function

$$R(s) = \frac{a_{21} + a_{22}\,s + a_{23}s^2 + \cdots \ldots\ldots a_{2r}s^{r-1}}{a_{11} + a_{12}\,s + a_{13}s^2 + \cdots \ldots\ldots + a_{1,r+1}s^r} \ldots\ldots\ldots (2)$$

Where r<nand $A_{i,j}$ and $a_{i,j}$ are scalar constants.Such that R(s) retains the important properties of G(s).

## REDUCTION METHOD

The $r^{th}$-order reduced approximant R(s) for G(s)is obtained by different methods as

# IMPROVED PADE APPROXIMATION USING STABILITY EQUATION

The reciprocal polynomial of D(s)be defined by

$$\widehat{D}(s) = s^n D(\frac{1}{s})$$

$$= A_{11}s^n + A_{12}s^{n-1} + \ldots\ldots\ldots\ldots + A_{1,n}s + A_{1,n+1} \;\ldots (3)$$

The reciprocal polynomial in eqn.3 has the property that it inverts the roots of the original polynomial and thus the small magnitude roots of D(s) will become the large magnitude roots of D(s) and vice versa. For stable G(s), the even and odd parts of D(s) may be factored as the following stability equations: [5]

$$D_e(s) = A_{11} \prod_{i=1}^{k_1} ( 1 + \frac{s^2}{z_i^2}) \ldots\ldots\ldots\ldots 4(a)$$

$$D_o(s) = A_{12}s \prod_{i=1}^{k_2} (1 + \frac{s^2}{p_i^2}) \ldots\ldots\ldots 4(b)$$

Where $k_1$ and $k_2$are the integer part of n/2 and (n — l)/2 respectively and $z_1^2 < p_1^2 < z_2^2 < p_2^2$... By discarding the factors with larger magnitudes of $z_i$and $p_i$ we have a reduced stability equation of order $r_1$ as

$$D_{r1}(s) = A_{11} \prod_{i=1}^{\frac{r1}{2}} (1 + \frac{s^2}{z_i^2}) + A_{12}s \prod_{i=1}^{\frac{r1-1}{2}} (1 + \frac{s^2}{p_i^2}) \ldots. (5)$$

As discussed above, only poles nearest to the origin are retained in $D_{r1}(s)$and no consideration is given to the poles which have large negative real parts. To ensure that $D_r(s)$also approximates some large magnitudepoles of G(s) stability equations similar to eqn. 4 are constructed for the reciprocal polynomial of eqn. 3 and a reduced polynomial $\widehat{D_{r2}}(s)$of order $r_2$is formed.$D_r(s)$is then found as

$$D_r(s) = D_{r1}(s). D_{r2}(s) \ldots\ldots\ldots. (6)$$

Where $D_{r2}(s)$ is the reciprocal polynomial of $\widehat{D_{r2}}(s)$ and$r = r_1 + r_2$.

Assuming that R(s)and G(s)have identical first α time moments and first β Markov parameters, the coefficients of numerator of reduced order model are then determined from [6]

$$a_{2,1} = a_{11}T_0$$

$$a_{2,2} = a_{11}T_1 + a_{12}T_0$$

.

.

$$a_{2,\alpha} = a_{11}T_{\alpha-1} + a_{12}T_{\alpha-2} + \ldots\ldots\ldots + a_{1,\alpha-1}T_1 + a_{1,\alpha}T_0$$

$$a_{2,(r-\beta+1)} = a_{1,r+1}M_{\beta-1} + a_{1,r}M_{\beta-2} + \ldots\ldots\ldots + a_{1,(r-\beta+3)}M_1 + a_{1,(r-\beta+2)}M_0$$

.

$$a_{2,r-1} = a_{1,r+1}M_1 + a_{1,r}M_0$$

$$a_{2,r} = a_{1,r+1}M_0$$

Where $\alpha + \beta = r.$

## CONTINUED FRACTION ALGORITHM

It is well known that the first *t* time moments and *(2r — t)* Markov parameters of *G(s)* are retained in the reduced modelformed by truncating the continued-fraction expansion after *2r*quotients, the first *t* quotients being formed by divisionfrom the constant terms and the next *(2r — t)* quotients by division from the highest powers of s. This gives the continued-fraction in the form

$$G(s) = \cfrac{1}{h1+\cfrac{s}{h_2+\cdots+\cfrac{s}{h_t+\cfrac{1}{sh_{t+1}+\cfrac{1}{h_{t+2}+\cdots+\cfrac{1}{sh_{2r-1}+\cfrac{1}{h_{2r}}}}}}}} \quad \ldots\ldots(7)$$

The $h_i$ *(i*= 1, 2 . . .*2r)* are readily obtained from the following **Routh-type array:**

| | | | |
|---|---|---|---|
| $h_1$ | $h_1$ | | $A_{11}$ $A_{12}$ $A_{13}$……… $A_{1,n+1}$ |
| | $h_2$ | | $A_{21}$ $A_{22}$ $A_{23}$ ……… $A_{2,n}$ |
| ⋮ | | | $A_{31}$ $A_{32}$ $A_{33}$ ……… $A_{3,n}$ |
| $h_t$ | ⋮ | | ⋮ |
| | $h_t$ | | $A_{t,1}$ $A_{t,2}$……… $A_{t,n+1-[t/2]}$ |
| | | | $A_{t+1,1}$ $A_{t+1,2}$ ……… $A_{t+1,n+1-[(t+1)/2]}$ |
| $h_{t+1}$ | | | $A_{t+2,1}$ $A_{t+2,2}$ ……… $A_{t+2,n+1-[(t+2)/2]}$ |
| $h_{t+2}$ | $h_{t+1}$ | | $B_{t+1,1}$ $B_{t+1,2}$……… $B_{t+1,n+1-[(t+1)/2]}$ |
| | $h_{t+2}$ | | $B_{t+2,1}$ $B_{t+2,2}$ ……… $A_{t+2,n+1-[(t+2)/2]}$ |
| ⋮ | | | $B_{t+3,1}$ $B_{t+3,2}$ ……… |
| $h_{2r-1}$ | ⋮ | | ⋮ |
| $h_{2r}$ | $h_{2r}$ | | $B_{2r,1}$ $B_{2r,2}$……… |
| | | | $B_{2r+1,1}$ $B_{2r+1,2}$ ……… |

Where

$h_i = A_{i,1}/A_{i+1,1}$ i = 1, 2… t

$B_{i,1}/B_{i+1,1}$ i = t+1… 2r

$$A_{i,j} = A_{i-2,j+1} - h_{i-2}A_{i-1,j+1}$$

$$i = 3,4, \ldots \ldots t+2, \quad j = 1,2, \ldots \ldots, n+1-[\frac{i}{2}]$$

$$B_{i,j} = B_{i-2,j+1} - h_{i-2}B_{i-1,j+1}$$

$$i = t+3, \ldots \ldots 2r+1, \quad j = 1,2, \ldots \ldots, n+1-[\frac{i}{2}]$$

And the last two rows of the $A_{i,j}$form the first two rows of the $B_{i,j}$by reversing the order of the elements, i.e.

$$B_{i,j} = A_{i,n+2-[\frac{i}{2}]-j}$$

$$i = t+1, t+2, \quad j = 1,2 \ldots, n+1-[\frac{i}{2}]$$

To invert the continued fraction (eqn. 7) an inverse array is formed with the same $h_i$($i$ = 1, 2...$2r$) starting from the bottom element $Q_{2r+1,1}$ (= 1, as this is arbitrary) which has the structure:

With

$$Q_{2r+1,1} = 1, Q_{i,1} = h_i Q_{i+1,1}$$

$$i = 2r, 2r-1, \ldots \ldots, t+1$$

$$a_{i,1} = h_i a_{i+1,1} \qquad i = t, t-1, \ldots \ldots, 1$$

$$Q_{i-2,j+1} = Q_{i,j} + h_{i-2}Q_{i-1,j+1}$$

$$i = 2r+1, \ldots \ldots, t+3, \quad j = 1,2, \ldots \ldots, r+1-[\frac{1}{2}]$$

$$a_{i-2,j+1} = a_{i,j} + h_{i-2}a_{i-1,j+1}$$

$$i = t+2, \ldots \ldots, 3 \quad j = 1,2, \ldots \ldots, r+1-[\frac{i}{2}]$$

And

$$a_{i,j} = Q_{i,r+2-[\frac{i}{2}]-j}$$

$$i = t+1, t+2, \quad j = 1,2, \ldots \ldots, r+1-[\frac{i}{2}]$$

The reduced transfer function is then given by the first two rows of the array (eqn.8).

## BIASEDMODEL REDUCTION BY FACTOR DIVISION

A reduced r[th]-order model is to be found such that

$$R(s) = \frac{N_r(s)}{D_r(s)} \qquad \ldots\ldots\ldots\ldots (9)$$

where $D_r(s)$ is the reduced stable denominator which may be found by one of the many techniques available and $N_r(s)$, the reduced numerator, isformed so that $R(s)$ retains the first $t$ time moments and $m$ Markov parameters of $G(s)$, where $t + m = r$.

Consider the polynomials defined by

$$D(s) = A_{11} + A_{12}s + \cdots \ldots \ldots \ldots . + A_{1,n+1}s^n$$

$$N_t(s) = A_{21} + A_{22}s + \cdots \ldots \ldots \ldots + A_{2,t-1}s^{t-1}$$

$$N_m(s) = A_{2,n-1}s^{n-1} + A_{2,n-2}s^{n-2} + \ldots \ldots \ldots + A_{2,n-m}s^{n-m}$$

$$n(s) = A_{1,t}s^t + A_{1,t+1}s^{t+1} + \ldots \ldots \ldots A_{1,n-m-1}s^{n-m-1}$$

Then from eqn.1

$$G(s) = \frac{N_t(s)}{D(s)} + \frac{N_m(s)}{D(s)} + \frac{n(s)}{D(s)} \qquad \ldots\ldots\ldots\ldots (10)$$

It is clear that only the expressions $N_t(s)/D(s)$ and $N_m(s)/D(s)$ contribute to the first $t$ time moments and m Markov parameters,respectively, to be retained in the reduced model.Thus $n(s)/D(s)$ may effectively be ignored for reduction purposesand eqn.10 is written as

$$G(s) = \frac{\frac{N_t(s)D_r(s)}{D(s)}}{D_r(s)} + \frac{\frac{N_m(s)D_r(s)}{D(s)}}{D_r(s)}$$

The reduced model is then given by

$$R(s) = \frac{\alpha_0 + \alpha_1 s + \cdots \ldots \ldots + \alpha_{t-1}s^{t-1}}{D_r(s)} + \frac{\beta_m s^{r-1} + \beta_{m-1}s^{r-2} + \ldots \ldots \ldots + \beta_1 s^t}{D_r(s)}$$

$$= \frac{\alpha_0 + \alpha_1 s + \cdots \ldots\ldots + \alpha_{t-1}s^{t-1} + \beta_1 s^t + \ldots\ldots\ldots + \beta_m s^{r-1}}{D_r(s)} \qquad \ldots\ldots\ldots (11)$$

Where $\frac{N_t(s)D_r(s)}{D(s)} = \alpha_0 + \alpha_1 s + \cdots \ldots \ldots + \alpha_{t-1}s^{t-1} \ldots \ldots$ is calculated by the factor division algorithm [8] and

$$\frac{N_m(s)D_r(s)}{D(s)} = \beta_m s^{r-1} + \beta_{m-1}s^{r-2} + \ldots\ldots\ldots + \beta_1 s^t + \ldots\ldots$$

Is calculated using division from the highest powers of s by a similar procedure, description is given in [9].

## SIMPLIFIED ROUTH APPROXIMATION METHOD

Let $A_{21}=B_0$, $A_{22}=B_1$……………$A_{2n}=B_{n-1}$

and$A_{11}=A_0$, $A_{12}=A_1$ …………...$A_{1,n+1}=A_n$

By using SRAM, the reduced denominator of r^th order model is defined as

$$D_r(s) = s^r + \frac{\alpha_r!}{A_0}\sum_{j=0}^{r-1} A_j s^j \ldots (12)$$

The α- parameters ($\alpha_1$, $\alpha_{2\ldots}$ $\alpha_r$) are obtained from the following simple Routh-type array:

**Alpha (α) table**

$$\alpha_1 = \frac{A_0}{A_1} \quad \begin{matrix} A_0 & A_1 & A_2 & A_3 & A_4 \ldots\ldots\ldots & A_n \\ A_1 & A_2 & A_3 & A_4 \ldots\ldots\ldots & A_n & \end{matrix}$$

$$\alpha_2 = \frac{C_1}{C_2} \quad \begin{matrix} C_1 & C_2 & C_3 & C_4 \ldots\ldots\ldots & C_n \\ C_2 & C_3 & C_4 & C_5 \ldots\ldots\ldots & C_n \end{matrix}$$

$$\alpha_3 = \frac{D_1}{D_2} \quad \begin{matrix} D_1 & D_2 & D_4 D_4 \ldots\ldots\ldots & D_{n-1} & \\ D_2 & D_3 & D_4 & D_5 \ldots\ldots\ldots & D_{n-1} \end{matrix}$$

$$\alpha_4 = \frac{E_1}{E_2} \quad \begin{matrix} E_1 & E_2 & E_3 & E_4 \ldots\ldots\ldots & E_{n-2} \\ E_2 & E_3 & E_4 & E_5 \ldots\ldots\ldots & E_{n-2} \end{matrix}$$

For i = odd

$C_i = A_i \qquad i = 1, 3, \ldots\ldots\ldots\ldots$

$D_1 = C_2$

$D_i = A_{i+1} \qquad i = 3, 5, \ldots\ldots\ldots\ldots$

$E_1 = D_2$

$E_i = A_{i+2} \qquad i = 3, 5, \ldots\ldots\ldots\ldots$

:

For i = even

$$\left.\begin{aligned} C_i &= A_i - \frac{A_0 A_{i+1}}{A_1} \\ D_i &= C_{i+1} - \frac{C_1 C_{i+2}}{C_2} \\ E_i &= D_{i+1} - \frac{D_1 D_{i+2}}{D_2} \end{aligned}\right\} i = 2, 4, 6 \ldots$$

:

Then for any given $D_r(s)$, the numerator $N_r(s)$ of the biased model, which will retain the first t time moments and m Markov parameters of G(s), is defined as

$$N_r(s) = N_{rt}(s) + N_{rm}(s) \qquad with\, r = t + m$$

$$= T_1 + T_2 s + \cdots \ldots \ldots .. + T_t s^{r-m+1} + M_m s^{r-m} \ldots \ldots \ldots \ldots .. + M_2 s^{r-2} + M_1 s^{r-1}$$

In general $T_t = \frac{a_{11}}{A_0} B_{t-1}$

And

$$M_m = \frac{1}{A_n}\left\{\sum_{i=1}^{m} B_{n-i}\, a_{1,r+1-(m-i)} - \sum_{j=0}^{m-1} M_j A_{n-(m-j)}\right\}$$

Where $M_0 = 0$

## EXAMPLE

Consider a 3rd order system also used in literature [6-7] which is reduced to the second order model by the above four methods

$$G(s) = \frac{8s^2 + 6s + 2}{s^2 + 4s^2 + 5s + 2}$$

By Applying Improved PadeApproximation Using Stability equation various second order reduced models are obtained as:

$$\alpha = \beta = 1, r_2 = 2:\; R_1(s) = \frac{8s + 5}{s^2 + 4s + 5}$$

$$\alpha = 2, r_2 = 2:\; R_2(s) = \frac{6.5s + 5}{s^2 + 4s + 5}$$

$$\alpha = 2, r_1 = r_2 = 1:\; R_3(s) = \frac{5.2s + 1.6}{s^2 + 4s + 5}$$

$$\alpha = 2, r_1 = 2: \; R_4(s) = \frac{1.5s + 0.5}{s^2 + 1.25s + 0.5}$$

Comparison of step responses of full system and reduced models is shown in figure1.

Applying continued fraction algorithm for biased model reduction we get two second order reduced model as

$$m = 1, t = 3: \quad R_5(s) = \frac{8s + 3.8976}{s^2 + 3.7338s + 3.8976}$$

$$m = 2, t = 2: \; R_6(s) = \frac{8s + 7.6}{s^2 + 4.2s + 7.6}$$

Comparison of step responses of full system and reduced models is shown in figure2.

Biased model reduction by factor division: Here denominator is obtained by Improved PadeApproximation Using Stability Equation as

$$D(s) = 4s^2 + 5s + 2$$

And α, β parameters are given as

$$\alpha_1 = 2, \alpha_2 = 6\,, \beta_1 = 32$$

Now two second order models are obtained as

$$m = 0, t = 2: \; R_7(s) = \frac{6s + 2}{4s^2 + 5s + 2}$$

$$m = 1, t = 1: \; R_8(s) = \frac{32s + 2}{4s^2 + 5s + 2}$$

Comparison of step responses of full system and reduced models is shown in figure3.

By SRAM: From α-table $\alpha_1 = 0.4$ and $\alpha_2 = 1.3888889$ and second order reduced denominator applying SRAM is obtained as

$$D(s) = s^2 + \frac{\alpha_1\alpha_2}{A_0}(A_0 + \; A_1 s)$$
$$= 0.5555556 + 1.3888889s + s^2$$

And$T_1 = 0.5555556$,$T_2 = 1.6666668$ and $M_1 = 8$

Two stable reduced models which retains t time moments and m Markov parameters are obtained from G(s), where t+m = 2

$$m = 0, t = 2: \; R_9(s) = \frac{1.6666668s + 0.5555556}{s^2 + 1.3888889s + 0.5555556}$$

$$m = 1, t = 1: \; R_{10}(s) = \frac{8s + 0.5555556}{s^2 + 1.3888889s + 0.5555556}$$

Comparison of step responses of full system and reduced models is shown in figure4.

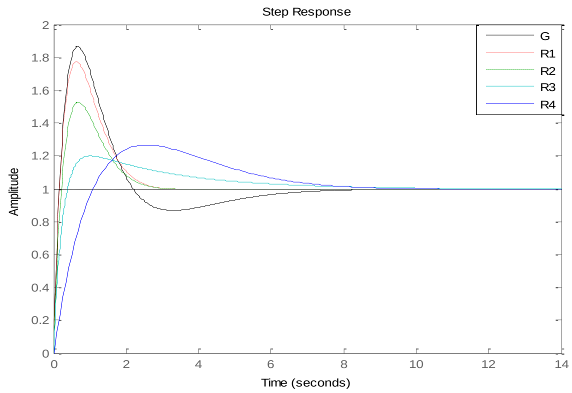


***Figure1-Comparison of step responses of G(s) and reduced models by Improved Pade Approximation Using Stability Equation***

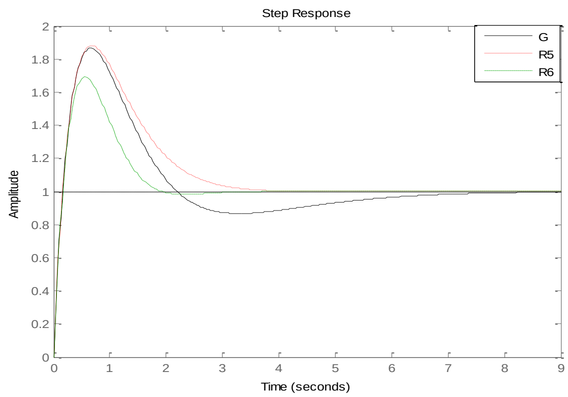


***Figure2-Comparison of step responses of G(s) and reduced models by Continued Fraction Algorithm***

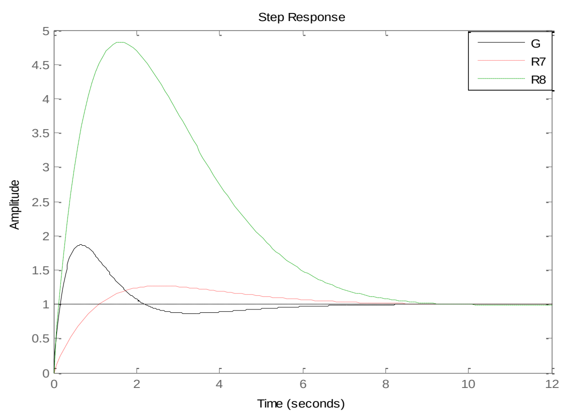


***Figure3-Comparison of step responses of G(s) and reduced models by Factor Division Algorithm***

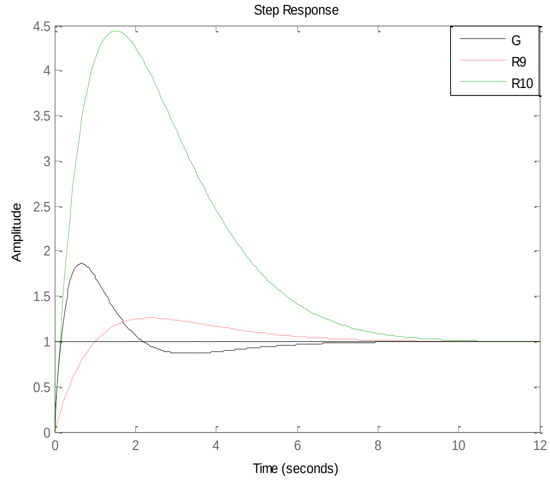


*Figure4-Comparison of step responses of G(s) and reduced models by SRAM*

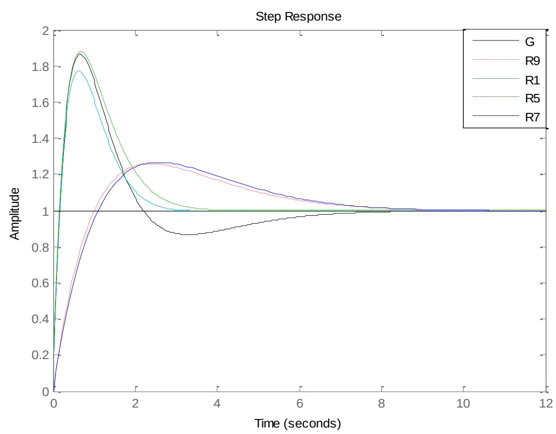


*Figure5-Comparison of step responses of G(s) and best responses given by different methods*

## CONCLUSIONS

It is shown that in model reduction by Pade Approximation, $R_1(s)$ shows the best overall time response approximation to G(s). In model reduction by Continued Fraction, $R_5(s)$ shows the best overall time response approximation to G(s). In model reduction by Factor Division, $R_7(s)$ shows the best overall time response approximation to G(s). In model reduction by SRAM, $R_8(s)$ shows the best overall time response approximation to G(s).The comparison of best responses of all the four methods and original system is shown in fig-5.And it is shown that model reduction by continued fraction $R_5(s)$ gives the best response. SRAM is the simplest method of model reduction because only one simple Routh-type array can be used to generate denominator as well as numerator of the reduced-order stable biased model. Table-1 gives the comparison on the basis of various parameters.

ISSN: 2277-9655
(I2OR), Publication Impact Factor: 3.785

## TABLE-1

| System Characteristics | | G(s) | $R_1(s)$ | $R_2(s)$ | $R_3(s)$ | $R_4(s)$ | $R_5(s)$ | $R_6(s)$ | $R_7(s)$ | $R_8(s)$ | $R_9(s)$ | $R_{10}(s)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Steady state | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Rise time(sec) | | 0.129 | 0.131 | 0.171 | 0.284 | 0.856 | 0.128 | 0.13 | 0.836 | 0.109 | 0.765 | 0.11 |
| Settling time(sec) | | 6.74 | 2.63 | 2.56 | 7.03 | 7.63 | 3.26 | 1.78 | 7.63 | 8.01 | 7.5 | 8.29 |
| Peak Response | Peak Amplitude | 1.87 | 1.73 | 1.53 | 1.2 | 1.27 | 1.88 | 1.69 | 1.27 | 4.82 | 1.26 | 4.44 |
| | Overshoot(%) | 86.5 | 77.3 | 52.7 | 19.9 | 26.6 | 88 | 69.1 | 26.6 | 382 | 26 | 344 |
| | At time(sec) | 0.656 | 0.63 | 0.682 | 1.02 | 2.6 | 0.674 | 0.55 | 2.6 | 1.59 | 2.34 | 1.51 |

## REFERENCES


1. SHAMASH, Y.: 'Model reduction using the Routh stability criterion and the Pade approximation techniques', *Int. J. Control,* 1975, 21, pp. 475-484.
2. HUTTON, M. F., and FRIEDLAND, B.: 'Routh approximations for reducing order of linear time-invariant systems', *IEEE Trans.,*
3. SHAMASH, Y.: 'Model reduction using the Routh stability criterionand the Pade approximation techniques', *Int. J. Control,* 1975, 21, pp. 475-484
4. CHEN, T. C. CHANG, C. Y', and HAN, K.W.: 'Stable reduced order pade approximants using stability equation method', *Electron. Lett.* 1980, 16, PP. 345-346.
5. PAL, J.:"ImprovedPade approximants using stability equation method", *Electron. Lett.* 1983, 19, pp. 426-427.
6. LUCAS, T.N.:"Continued fraction algorithm for biased model reduction", ibid.1983, Vol.19, No.-12, PP. 444-445.
7. LUCAS, T. N.: 'Factor division: a useful algorithm in model reduction', *IEE Proc. D, Control Theory &Appl,* 1983, 130**,** pp. 362-364
8. LUCAS, T.N.:" Biased model reduction by factor division", ibid. 1984, 20, pp.-582-583.
9. SASTRY, G.V.K.R., and KRISHNAMURTHY, V: "Biased model reduction by simplified RouthApproximation method, Electronics letters, 24th Sept. 1987, Vol. 23, No.-20, and PP. 1045-1047.
10. Prasad R, S. P. Sharma and A. K. Mittal, 'Improved Pade approximants for multivariable systems using stability equation method', *J. Institution of Engineers India IE( I) Journal-EL,* Vol. 84, pp. 161-165 Jan. 2003.